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Best bounds on the approximation of polynomials and splines by their control structure

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Abstract

We present best bounds on the deviation between univariate polynomials, tensor product polynomials, Bézier triangles, univariate splines, and tensor product splines and the corresponding control polygons and nets. Both pointwise estimates and bounds on the L_p-norm are given in terms of the maximum of second differences of the control points. The given estimates are sharp for control points corresponding to arbitrary quadratic polynomials in the univariate case, and to special quadratic polynomials in the bivariate case. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Spline control polygons and nets model the functions they represent, and the order of convergence is well known as the knot spacing tends to zero (Cohen and Schumaker, 1985). First quantitative bounds better than those implied by the convex hull property are given in (Prautzsch and Kobbelt, 1994). However, they are not optimal and hard to evaluate since they depend on derivatives of second through highest order of the given spline or Bézier curve. Then recently, Nairn, Peters, ad Lutterkort published a pioneering paper on polynomials in Bernstein–Bézier form (Nairn et al., 1999). For various values of p and μ, they succeed in specifying *best constants* $N(d, p, \mu)$ such that the maximal deviation between a polynomial and its Bézier control polygon is bounded by $N(d, p, \mu)\|\Delta^\mu P\|_p$, where $\Delta^\mu P$ denotes the vector of μth forward differences of the control points P.

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Sharp results on uniform splines were first presented in a conference talk by David Lutterkort.[2] Meanwhile, non-uniform splines have been discussed in (Lutterkort and Peters, 1999), where the difference between a spline and its control polygon is bounded by a piecewise linear function times the maximum of certain weighted second differences of the control points. For specific convex splines, this bound is sharp at the Greville abscissae.

The approach presented here is different, and was developed at the same time and independent of (Lutterkort and Peters, 1999). Using only basic spline properties like linear precision and positivity, it is completely elementary and admits to construct a bounding function which is *sharp everywhere*. For a spline $B^d(t)P$ of degree d with control points P denote the control polygon by $H^d(t)P$. Further, let $B^d(t)P^* = t^2/2$. Then it is our main result that

$$\left|B^d(t)P - H^d(t)P\right| \leqslant \left|B^d(t)P^* - H^d(t)P^*\right| \|\Delta^2 P\|_\infty,$$

where $\Delta^2 P$ denotes the vector of control points of the second derivative $D^2 B^d(t)P = B^{d-2}(t)\Delta^2 P$. With μ_j the *mean value*, i.e., a Greville abscissa, and σ_j^2 the *variance* of always d consecutive knots, we obtain in particular

$$\left|B^d(\mu_j)P - H^d(\mu_j)P\right| \leqslant \frac{\sigma_j^2}{2d} \|\Delta^2 P\|_\infty,$$

and

$$\left\|B^d P - H^d P\right\|_\infty \leqslant \frac{\sup_j \sigma_j^2}{2d} \|\Delta^2 P\|_\infty$$

implying the quadratic convergence of the control polygon as the knot spacing tends to zero. In all three estimates, equality holds if $B^d P$ is a quadratic polynomial.

Evidently, the pointwise estimate implies bounds on L_p-norms of the difference. Further, we generalize our results to tensor product splines and Bézier triangles. While general L_p-results are interesting in their own right, L_∞ bounds are also useful in applications such as intersection or rendering algorithms.

The paper is organized as follows: In the next section, we detail the basic ideas of our approach for the special case of polynomials in Bernstein–Bézier form. Further, we derive a lower bound showing that the given upper bound is asymptotically exact when applied in the context of subdivision. The third and the fourth section generalize the univariate results to bivariate tensor product polynomials and Bézier triangles. Finally, in the fifth section we present local and global results for univariate splines and tensor product splines with arbitrary knot sequences.

2. Polynomials in Bernstein–Bézier form

Let $\mathbb{P}^d$ be the space of polynomials of degree less than or equal to d on the unit interval $[0, 1]$. We denote by

[2] D. Lutterkort, 11/16/1998, Dagstuhl conference on “CAD-Tools and Algorithms for Product Design”.

$$B^d = \left[B_0^d, \dots, B_d^d\right], \qquad H^d = \left[H_0^d, \dots, H_d^d\right],$$
$$L^d = \left[L_0^d, \dots, L_d^d\right] = B^d - H^d$$

the row vectors of the Bernstein polynomials of degree d, the hat functions with respect to the break points $t_j^d = j/d$, $j = 0, \dots, d$, and their differences, respectively. Any polynomial $g \in \mathbb{P}^d$ can be written in Bernstein–Bézier form as $g = B^d P$, where $P = [P_0, \dots, P_d]^{\mathrm{t}} \in \mathbb{R}^{d+1}$ is a column vector of real valued control points. The corresponding control polygon $h = H^d P$ is a piecewise linear function with values P_j at the break points t_j^d. Further, we denote by $\Delta^\mu P = [\Delta_0^\mu P, \dots, \Delta_{d-\mu}^\mu P]^{\mathrm{t}}$ the vector of the μth forward differences of P.

2.1. A local bound

It is our goal to derive a sharp bound on the deviation $|L^d P|$ between a polynomial $B^d P$ and its control polygon $H^d P$ at a fixed argument $t \in [t_k^d, t_{k+1}^d] \subset [0, 1]$. More precisely, we are seeking the smallest constant $N(t, d)$ with

$$\left|\left(B^d(t) - H^d(t)\right)P\right| = \left|L^d(t)P\right| \leqslant N(t, d)\|\Delta^2 P\|_\infty$$

for all P. We note that $\Delta^2(P - P') = 0$ if and only if $P - P'$ is a linear sequence of the form $\alpha i + \beta$, $i = 0, \dots, d$, what in turn is equivalent to $L^d P = L^d P'$ by linear precision. This observation has two consequences. First, we can assume $P_k = P_{k+1} = H^d(t)P = 0$ without loss of generality when considering the range of $L^d(t)P$. Second, since the above inequality is trivial for $\|\Delta^2 P\|_\infty = 0$, we can scale it and obtain

$$N(t, d) := \max_{P \in \mathbb{R}_k^{d+1}} \left|B^d(t)P\right|, \quad \mathbb{R}_k^{d+1} := \left\{P \in \mathbb{R}^{d+1}\colon\ P_k = P_{k+1} = 0 \wedge \|\Delta^2 P\|_\infty = 1\right\}.$$

It is quite evident that the maximum is attained if all second differences of P are as large as possible. To be more specific, consider a vector Q with $Q_k = Q_{k+1} = 0$ and $\Delta^2 Q \geqslant 0$. The corresponding control polygon $H^d Q$ is convex, hence $Q \geqslant 0$ and also $B^d(t)Q \geqslant 0$. Let $P^k \in \mathbb{R}_k^{d+1}$ satisfy $\Delta^2 P^k = [1, \dots, 1]^{\mathrm{t}}$. Then $\Delta^2 P^k \pm \Delta^2 P = \Delta^2(P^k \pm P) \geqslant 0$ and consequently $B^d(t)(P^k \pm P) = B^d(t)P^k \pm B^d(t)P \geqslant 0$ for all $P \in \mathbb{R}_k^{d+1}$. Thus, $B^d(t)P^k \geqslant |B^d(t)P|$ and $N(t, d) = B^d(t)P^k$ is the wanted constant for $t \in [t_k^d, t_{k+1}^d]$. More generally, by linear precision it is $N(t, d) = L^d(t)P$ for any P with $\Delta^2 P = [1, \dots, 1]^{\mathrm{t}}$. For instance, we may choose $P_j := j(j - d)/2$ independent of k.

To sum up, we obtain the following result on the difference $L^d P$ between a polynomial $B^d P$ and its control polygon $H^d P$:

Theorem 2.1. *Let $P_j^* := j(j - d)/2$, $j = 0 : d$. Then*

$$\left|L^d(t)P\right| \leqslant L^d(t)P^*\|\Delta^2 P\|_\infty$$

for all $t \in [0, 1]$ and $P \in \mathbb{R}^{d+1}$. Equality holds if $B^d P \in \mathbb{P}^2$.

The polynomial $q^* := B^d P^*$ is quadratic and has zeros at $t \in \{0, 1\}$. Since $D^2 q^* = d(d - 1)$, we obtain $q^*(t) = d(d - 1)t(t - 1)/2$. The control polygon $H^d P^*$ can be evaluated easily, but also the following estimate might be of interest. Let $q^+(t) = d^2t(t-1)/2$ be the parabola interpolating the control polygon at the break points. Then $q^+(t) \leqslant H^d(t)P^*$ for $t \in [0,1]$ and we obtain

$$L^d(t)P^* \leqslant q^*(t) - q^+(t) = dt(1-t)/2 =: \widetilde{N}(t,d).$$

The sub-optimal constant $\widetilde{N}(t,d)$ is in general slightly worse than $N(t,d)$, and only sharp for $t = t_j^d$ being a break point. However, the resulting inequality takes on a very simple form,

$$\left|L^d(t)P\right| \leqslant \frac{dt(1-t)}{2}\|\Delta^2 P\|_\infty.$$

2.2. Global bounds

The results of the previous section immediately imply bounds on L_p-norms since for fixed $\|\Delta^2 P\|_\infty$ the pointwise deviations are maximized simultaneously by quadratic polynomials with $q^* = B^d P^*$ being a canonical representative. The following result contains Theorem 3.1 and Corollary 3.1 in (Nairn et al., 1999) as special cases.

Theorem 2.2. *Denote by $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$ the L_p-norm on the unit interval, and let $P_j^* = j(j-d)/2$, $j = 0:d$. Then*

$$\|L^d P\|_p \leqslant \|L^d P^*\|_p \|\Delta^2 P\|_\infty$$

for all $P \in \mathbb{R}^{d+1}$, where equality holds if $B^d P \in \mathbb{P}^2$.

With $\omega_2(d) = 0$ if d is even, and $\omega_2(d) = 1$ if d is odd, we obtain for instance

$$\|L^d P^*\|_1 = \frac{d-1}{12}, \qquad \|L^d P^*\|_2 = \left(\frac{3d^3 - 5d^2 + 3d - 1}{360d}\right)^{1/2},$$
$$\|L^d P^*\|_\infty = \frac{d}{8} - \frac{\omega_2(d)}{8d}.$$

Finding the first two values amounts to summing up finite quadratic and cubic sequences, whereas the third one can be determined following an argument from (Nairn et al., 1999). $L^d P^* = |L^d P^*|$ is a piecewise convex function, hence attains its maximum at one of the break points. For $t = t_j^d$ we have $L^d(t)P^* = dt(1-t)/2$. Hence, evaluation at the break point $t^d_{\lfloor d/2 \rfloor} = 1/2 - \omega_2(d)/2d$, which is the nearest neighbor of the center $t = 1/2$, yields the given constant. An explicit bound in closed form can be found when estimating $L^d(t)P^*$ by the sub-optimal constant $\widetilde{N}(t,d)$,

$$\|L^d P^*\|_p \leqslant \frac{d}{2}\left\|t(1-t)\right\|_p = \frac{d}{2}\left(\frac{\Gamma(p+1)^2}{\Gamma(2p+2)}\right)^{1/p}.$$

We obtain for instance

$$\|L^d P^*\|_1 \leqslant d/12, \qquad \|L^d P^*\|_2 \leqslant d/\sqrt{120}, \qquad \|L^d P^*\|_\infty \leqslant d/8.$$

2.3. A lower bound

Let $g_{[a,b]}(t) := g(a+t(b-a)) = B^d P_{[a,b]}$ be the Bernstein–Bézier representation of the part of the polynomial $g = B^d P$ corresponding to the interval $[a,b] \subset [0,1]$. Subdivision, which refers to splitting g into two parts $g_{[0,x]}$, $g_{[x,1]}$ at a point $x \in (0,1)$, is a convenient tool, e.g., for adaptive rendering of Bézier curves.

In order to optimally reduce the deviation between a control polygon and the corresponding polynomial it is suggested in (Nairn et al., 1999) to choose x so that the maximum of $\|\Delta^2 P_{[0,x]}\|_\infty$ and $\|\Delta^2 P_{[x,1]}\|_\infty$ is minimized. However, sharpness of the bound provided by Theorem 2.1 is not sufficient to justify this approach rigorously. The crucial point is that the actual error is not only estimated, but in fact approximated by the given bound. The following lower bound is in general not sharp, but helpful for judging this problem.

Theorem 2.3. *With the notations of Theorem* 2.1 *we have*

$$\|L^d P^*\|_p\big(\|\Delta^2 P\|_\infty - \|\Delta^3 P\|_1\big) \leqslant \|L^d P\|_p \leqslant \|L^d P^*\|_p \|\Delta^2 P\|_\infty, \quad P \in \mathbb{R}^{d+1}.$$

Proof. Without loss of generality, we assume $\|\Delta^2 P\|_\infty = \max_j \Delta_j^2 P = 1$. The inverse triangle inequality implies

$$\|L^d P\|_p \geqslant \|L^d P^*\|_p - \big\|L^d(P-P^*)\big\|_p \geqslant \|L^d P^*\|_p\big(1 - \|\Delta^2(P-P^*)\|_\infty\big).$$

Since $\Delta^2 P^* = 1$, the vector $\Delta^2(P-P^*)$ contains a zero. Hence, it can be estimated by $\|\Delta^2(P-P^*)\|_\infty \leqslant \|\Delta^3(P-P^*)\|_1 = \|\Delta^3 P\|_1$ and the assertion follows. □

The inclusion of the actual deviation provided by Theorem 2.3 is asymptotically convergent in the following sense. Consider the segment $g_{[a,b]} = B^d P_{[a,b]}$ of the polynomial $g = B^d P$. With $h := b-a$ we have $\Delta^3 P_{[a,b]} = \mathrm{O}(h^3)$ as $h \to 0$ implying that the difference $L^d P^* \|\Delta^3 P_{[a,b]}\|_1$ between the upper and the lower bound is also of order $\mathrm{O}(h^3)$. On the other hand, the deviation between the control polygon and the corresponding polynomial decays as $\mathrm{O}(h^2)$, and not faster than that away from zeros of the second derivative. Hence, the ratio of the actual deviation $\|L^d P_{a,b}\|_p$ and the bound $\|L^d P^*\|_p \|\Delta^2 P_{[a,b]}\|_\infty$ tends to 1 as $h \to 0$ almost everywhere. We conclude that at least asymptotically the given bound is indeed well suited for predicting optimal subdivision points.

3. Tensor product polynomials

The results of the preceding section readily generalize to the tensor product setting. Confining ourselves to the bivariate case for the sake of simplicity, we consider the space $\mathbb{P}^{\mathbf{d}}$ of polynomials of bi-degree $\mathbf{d} = (d_1, d_2)$. Any polynomial $g \in \mathbb{P}^{\mathbf{d}}$ can be written in Bernstein–Bézier form as $g(t_1,t_2) = B^{d_1}(t_1) P (B^{d_2}(t_2))^{\mathrm{t}}$, where P is a $(d_1+1)\times(d_2+1)$-matrix of control points. The control net corresponding to $g(t_1,t_2)$ is the piecewise bi-linear function $h(t_1,t_2)=H^{d_1}(t_1)P(H^{d_2}(t_2))^{\mathrm t}$. The deviation between g and h is expressed by means of a linear operator $\mathbf{L}^{\mathbf d}(t_1,t_2)$ acting on matrices of control points according to

$$\mathbf{L}^{\mathbf d}(t_1,t_2)P := B^{d_1}(t_1)P\bigl(B^{d_2}(t_2)\bigr)^{\mathrm t} - H^{d_1}(t_1)P\bigl(H^{d_2}(t_2)\bigr)^{\mathrm t}.$$

The vector $L^d(t)=B^d(t)-H^d(t)$ continues to denote the difference vector of univariate Bernstein polynomials and hat functions.

In order to measure directional second forward differences, we define

$$\|\Delta_1^2 P\|_\infty := \max_{0\leqslant i\leqslant d}\|\Delta^2 P_i\|_\infty, \qquad \|\Delta_2^2 P\|_\infty := \|\Delta_1^2 P^{\mathrm t}\|_\infty,$$

where $P_0,\dots,P_d$ are the column vectors of P.

3.1. Local and global bounds

Theorem 2.1 implies the local bound

$$\begin{aligned}|\mathbf{L}^{\mathbf d}(t_1,t_2)P| &= \bigl|B^{d_1}(t_1)P\bigl(B^{d_2}(t_2)\bigr)^{\mathrm t} - H^{d_1}(t_1)P\bigl(H^{d_2}(t_2)\bigr)^{\mathrm t}\bigr|\\ &\leqslant \bigl|L^{d_1}(t_1)P\bigl(H^{d_2}(t_2)\bigr)^{\mathrm t}\bigr| + \bigl|B^{d_1}(t_1)P\bigl(L^{d_2}(t_2)\bigr)^{\mathrm t}\bigr|\\ &\leqslant L^{d_1}(t_1)P^*\|\Delta_1^2P\|_\infty + L^{d_2}(t_2)P^*\|\Delta_2^2P\|_\infty.\end{aligned}$$

Equality does not hold for all bi-quadratic polynomials, but for polynomials of the form $a_1t_1^2+a_2t_2^2$ with $a_1a_2\geqslant 0$. For the L_p-norm of the deviation we obtain

$$\|\mathbf{L}^{\mathbf d}P\|_p \leqslant \bigl\|L^{d_1}(t_1)P^*\|\Delta_1^2P\|_\infty + L^{d_2}(t_2)P^*\|\Delta_2^2P\|_\infty\bigr\|_p.$$

Evaluation of the right hand side yields for instance

$$\begin{aligned}\|\mathbf{L}^{\mathbf d}P\|_1 &\leqslant \|L^{d_1}P^*\|_1\|\Delta_1^2P\|_\infty + \|L^{d_2}P^*\|_1\|\Delta_2^2P\|_\infty,\\ \|\mathbf{L}^{\mathbf d}P\|_2 &\leqslant \left(\sum_{i=1}^2\|L^{d_i}P^*\|_2^2\|\Delta_i^2P\|_\infty^2 + 2\prod_{i=1}^2\|L^{d_i}P^*\|_1\|\Delta_i^2P\|_\infty\right)^{1/2},\\ \|\mathbf{L}^{\mathbf d}P\|_\infty &\leqslant \|L^{d_1}P^*\|_\infty\|\Delta_1^2P\|_\infty + \|L^{d_2}P^*\|_\infty\|\Delta_2^2P\|_\infty.\end{aligned}$$

For general p, evaluation of the bound is costly since it does not reduce to precomputing constants. Instead, integration has to be performed repeatedly for varying control points. In this case, the estimate

$$\|\mathbf{L}^{\mathbf d}P\|_p \leqslant \|L^{d_1}P^*\|_p\|\Delta_1^2P\|_\infty + \|L^{d_2}P^*\|_p\|\Delta_2^2P\|_\infty$$

provided by the triangle inequality in L_p-spaces can be used.

4. Bézier triangles

Let $\mathbb{P}^d$ be the space of bivariate polynomials of total degree less than or equal to d on the unit triangle $T := \{(u,v)\in\mathbb{R}_0^+\times\mathbb{R}_0^+\colon\ u+v\leqslant 1\}$. The set of multi-indices with norm d is denoted by $I^d := \{\boldsymbol\alpha\in\mathbb{N}_0^3\colon\ |\boldsymbol\alpha| = \boldsymbol\alpha_1+\boldsymbol\alpha_2+\boldsymbol\alpha_3 = d\}$ and contains $n(d) := (d+1)(d+2)/2$ elements. Using for instance lexicographical ordering, we collect the

Bernstein polynomials $B^d_{\boldsymbol\alpha}$, $\boldsymbol\alpha \in I^d$, of degree d in a row vector B^d and the corresponding control points $P_{\boldsymbol\alpha}$ in a column vector $P \in \mathbb{R}^{n(d)}$. Then any bivariate polynomial $g \in \mathbb{P}^d$ can be written in Bernstein–Bézier form as $g = B^d P$.

The three direction mesh with vertices $\mathbf{t}^d_{\boldsymbol\alpha} := (\boldsymbol\alpha_1/d, \boldsymbol\alpha_2/d)$, $\boldsymbol\alpha \in I^d$, provides a triangulation of T. Its triangles $T_{\boldsymbol\alpha,\boldsymbol\beta,\boldsymbol\gamma} := \operatorname{conv}(\mathbf{t}^d_{\boldsymbol\alpha}, \mathbf{t}^d_{\boldsymbol\beta}, \mathbf{t}^d_{\boldsymbol\gamma})$ are the convex hulls of always three neighboring vertices with indices $|\boldsymbol\alpha - \boldsymbol\beta| = |\boldsymbol\alpha - \boldsymbol\gamma| = |\boldsymbol\beta - \boldsymbol\gamma| = 1$. The hat functions $H^d_{\boldsymbol\alpha}$, $\boldsymbol\alpha \in I^d$, with respect to this triangulation are collected in a row vector H^d, and the control net corresponding to the polynomial $B^d P$ is the piecewise linear function $H^d P$ with values $P_{\boldsymbol\alpha}$ at the vertices $\mathbf{t}^d_{\boldsymbol\alpha}$. The difference between Bernstein polynomials and hat functions is denoted by $L^d := B^d - H^d$.

With the directions $\boldsymbol\delta_1 = (0, 1, -1)$, $\boldsymbol\delta_2 = (-1, 0, 1)$, $\boldsymbol\delta_3 = (1, -1, 0)$, the mixed second differences at the inner control points are

$$\Delta^2_{\boldsymbol\alpha,i} P := P_{\boldsymbol\alpha-\boldsymbol\delta_{i+1}} + P_{\boldsymbol\alpha-\boldsymbol\delta_{i+2}} - P_{\boldsymbol\alpha+\boldsymbol\delta_i} - P_{\boldsymbol\alpha}, \quad i = 1, 2, 3 \quad \bmod 3.$$

They are collected in a vector $\Delta^2 P$ and measure the kink between neighboring facets of the control net in such a way that the control polygon is convex if and only if $\Delta^2 P \geqslant 0$.

4.1. A local bound

Let $(u, v) \in T_{\boldsymbol\alpha,\boldsymbol\beta,\boldsymbol\gamma} \subset T$ be fixed, then we are seeking the smallest constant $N(u, v, d)$ with

$$\left|L^d(u, v)P\right| \leqslant N(u, v, d)\|\Delta^2 P\|_\infty$$

for all P. As in the univariate case, we infer from linear precision that $\Delta^2(P - P') = 0$ if and only if $L^d P = L^d P'$, hence

$$N(u, v, d) = \max\left\{|B^d P|\colon\ P_{\boldsymbol\alpha} = P_{\boldsymbol\beta} = P_{\boldsymbol\gamma} = 0 \wedge \|\Delta^2 P\|_\infty = 1\right\}.$$

Using the same convexity and positivity arguments as in Section 2, we find that the maximum is attained if all mixed second differences are maximal. In general, we have $N(u, v, d) = L^d P$ for any P with $\Delta^2 P = [1, \ldots, 1]^{\mathrm t}$. Again, a suitable control net P^* can be found independent of the location of (u, v). The following choice is easily verified by inspection.

Theorem 4.1. *Let $P^*_{\boldsymbol\alpha} = -(\boldsymbol\alpha_1\boldsymbol\alpha_2 + \boldsymbol\alpha_1\boldsymbol\alpha_3 + \boldsymbol\alpha_2\boldsymbol\alpha_3)$, $\boldsymbol\alpha \in I^d$. Then*

$$\left|L^d(u, v)P\right| \leqslant L^d(u, v)P^*\|\Delta^2 P\|_\infty$$

for all $(u, v) \in T$ and $P \in \mathbb{R}^{n(d)}$.

Equality does not hold for all $B^d P \in \mathbb{P}^2$, but for $B^d P \in \mathbb{P}^1 \oplus \operatorname{span}\{B^d P\}$. The control net P^* yields the polynomial

$$q^*(u, v) := B^d(u, v)P^* = d(1 - d)(uv + vw + wu), \quad w := 1 - u - v,$$

while the quadratic polynomial interpolating the control net at the vertices is $q^+(u, v) := -d^2(uv + vw + wu)$. By $q^+(u, v) \leqslant H^d(u, v)P^*$, we obtain

$$L^d(u, v)P^* \leqslant q^*(u, v) - q^+(u, v) = d(uv + vw + wu) =: \widetilde{N}(u, v, d),$$

and the resulting inequality is

$$\left|L^d(u,v)P\right| \leqslant d(uv+vw+wu)\|\Delta^2 P\|_\infty.$$

4.2. Global bounds

The pointwise estimate of Theorem 4.1 implies the following global bounds.

Theorem 4.2. *For $1 \leqslant p \leqslant \infty$ denote by $\|\cdot\|_p$ the L_p-norm on the unit triangle T, and let $P^*_{\boldsymbol\alpha} = -(\boldsymbol\alpha_1\boldsymbol\alpha_2 + \boldsymbol\alpha_1\boldsymbol\alpha_3 + \boldsymbol\alpha_2\boldsymbol\alpha_3)$, $\boldsymbol\alpha \in I^d$. Then*

$$\|L^d P\|_p \leqslant \|L^d P^*\|_p \|\Delta^2 P\|_\infty$$

for all $P \in \mathbb{R}^{n(d)}$.

With $\omega_3(d) = 0$ if d is a multiple of 3, and $\omega_3(d) = 1$ otherwise, we obtain for instance

$$\|L^d P^*\|_1 = \frac{d-1}{8}, \quad \|L^d P^*\|_2 = \left(\frac{8d^3 - 15d^2 + 8d - 1}{240d}\right)^{1/2},$$
$$\|L^d P^*\|_\infty = \frac{d}{3} - \frac{\omega_3(d)}{3d}.$$

Again, the first two values are obtained by summing up finite sequences. For determining the third one, we note that $L^d P^* = |L^d P^*|$ is a piecewise convex function, which attains its maximum at one of the vertices. For $(u,v) = \mathbf{t}^d_{\boldsymbol\alpha}$ we have $L^d(u,v)P^* = \widetilde{N}(u,v,d)$. Let $d = 3m + k$, $k \in \{-1, 0, 1\}$. Then $\mathbf{t}_{(m,m,d-2m)}$ is the nearest neighbor of the center $(u,v) = (1/3, 1/3)$, and $\widetilde{N}(m/d, m/d)$ is the given constant. Simplified expressions are obtained when estimating $L^d(u,v)P^*$ by the sub-optimal constant $\widetilde{N}(u,v,d)$,

$$\|L^d P^*\|_p \leqslant d\|uv + vw + wu\|_p.$$

We obtain for instance

$$\|L^d P^*\|_1 \leqslant d/8, \qquad \|L^d P^*\|_2 \leqslant d/\sqrt{30}, \qquad \|L^d P^*\|_\infty \leqslant d/3.$$

5. Splines

For $d \geqslant 2$ let $\mathbb{S}^d_T$ be the space of splines of degree $\leqslant d$ with knot sequence $T = \{t_j\}_{j\in\mathbb{Z}}$, where we assume continuity throughout, i.e., all knots have multiplicity $\leqslant d$. We denote by $B^d = \{B^d_j\}_{j\in\mathbb{Z}}$ the sequence of B-splines with supp $B^d_j = [t_j, t_{j+d+1}]$. Any spline $g \in \mathbb{S}^d_T$ can be written as $g = B^d P := \sum_{j\in\mathbb{Z}} B^d_j P_j$ with $P \in \mathbb{R}^{\mathbb{Z}}$ a sequence of real control points.

Mean value and variance of d consecutive knots are given by

$$\mu_j := \frac{1}{d}\sum_{i=1}^{d} t_{j+1}, \qquad \sigma_j^2 := \frac{1}{d-1}\sum_{i=1}^{d}(t_{j+i} - \mu_j)^2.$$

The points μ_j, also referred to as the Greville abscissae of T, form a strictly monotone increasing sequence $M = \{\mu_j\}_{j\in\mathbb{Z}}$. The hat functions with respect to M are denoted by

$H^d = \{H_j^d\}_{j\in\mathbb{Z}}$. The control polygon $h = H^d P := \sum_{j\in\mathbb{Z}} H_j^d P_j$ associated with the spline $B^d P$ is a piecewise linear function with values P_j at μ_j. The difference between B-splines and hat functions is denoted by $L^d := B^d - H^d$.

Further, we define the difference operators $\Delta^\mu : P \mapsto \{\Delta_j^\mu P\}_{j\in\mathbb{Z}}$ by $D^\mu B^d P = B^{d-\mu}\Delta^\mu P$. In particular,

$$\Delta_j^1 P = d\,\frac{P_j - P_{j-1}}{t_{j+d} - t_j}, \qquad \Delta_j^2 P = (d-1)\frac{\Delta_j^1 P - \Delta_{j-1}^1 P}{t_{j+d-1} - t_j}.$$

$\Delta_j^1 P$ is the slope of the control polygon on $[\mu_{j-1}, \mu_j]$, while $\Delta_j^2 P$ is a positive multiple of the second divided difference of $H^d P$ at the Greville abscissae $[\mu_{j-2}, \mu_{j-1}, \mu_j]$. Thus, $\Delta^2 P \geqslant 0$ if and only if the control polygon is convex.

5.1. A local bound

Let $t \in [\mu_k, \mu_{k+1}]$ be fixed. Then we are seeking the smallest constant $N(t,d)$ with

$$\left|\big(B^d(t) - H^d(t)\big)P\right| = \left|L^d(t)P\right| \leqslant N(t,d)\|\Delta^2 P\|_\infty$$

for all P. As before, we infer from linear precision that $\Delta^2(P - P') = 0$ if and only if $L^d P = L^d P'$, hence

$$N(t,d) := \max_{P\in\mathbb{R}_k^{\mathbb{Z}}} \left|B^d(t)P\right|, \qquad \mathbb{R}_k^{\mathbb{Z}} := \left\{P \in \mathbb{R}^{\mathbb{Z}}:\ P_k = P_{k+1} = 0 \wedge \|\Delta^2 P\|_\infty = 1\right\}.$$

Using the same convexity and positivity arguments as above, we find that the maximum is attained if all second differences are maximal. In general, we have $N(t,d) = L^d P$ for any P with $\Delta_j^2 P = 1$ for all $j \in \mathbb{Z}$. Again, a suitable control polygon can be chosen independent of the location of t. In order to do so, we consider Marsden's identity,

$$(t-\tau)^d = \sum_{j\in\mathbb{Z}} B_j^d(t) \prod_{i=j+1}^{j+d} (t_i - \tau).$$

Differentiating $(d-2)$-times with respect to τ, we find that the sequence

$$P_j^* := \frac{1}{d(d-1)} \sum_{i=j+1}^{j+d} \sum_{\ell=i+1}^{j+d} t_i t_\ell = \frac{\mu_j^2}{2} - \frac{\sigma_j^2}{2d}, \quad j \in \mathbb{Z}, \tag{5.1}$$

generates the spline $B^d(t)P^* = t^2/2$. In particular, $\Delta_j^2 P^* = 1$ for all $j \in \mathbb{Z}$.

Theorem 5.1. *With P^* defined according to* (5.1),

$$\left|L^d(t)P\right| \leqslant L^d(t)P^*\|\Delta^2 P\|_\infty$$

for all $t \in \mathbb{R}$ and $P \in \mathbb{R}^{\mathbb{Z}}$. Equality holds if $B^d P \in \mathbb{P}^2$.

Evidently, it is possible to restrict the norm on the right hand side to those entries of $\Delta^2 P$ which depend on control points affecting $L^d(t)P$. More precisely, let $t \in [t_k, t_{k+1})$. Since $\operatorname{supp} B_j^d = [t_j, t_{j+d+1}]$ and $\mu_{k-d} \leqslant t_k < t_{k+1} \leqslant \mu_k$,

$$B^d(t)P = \sum_{j=k-d}^{k} B_j^d(t)P_j, \qquad H^d(t)P = \sum_{j=k-d}^{k} H_j^d(t)P_j.$$

The control points $P_{k-d}, \ldots, P_k$ define the second differences $\Delta^2_{k-d+2}P, \ldots, \Delta^2_k P$. Hence, we set

$$\|\Delta^2 P\|_{t,\infty} := \max\bigl\{|\Delta^2_k P|:\ k-d+2 \leqslant j \leqslant k,\ t \in [t_k, t_{k+1})\bigr\},$$

and specify the statement of Theorem 5.1 by

$$\bigl|L^d(t)P\bigr| \leqslant L^d(t)P^* \|\Delta^2 P\|_{t,\infty}, \quad t \in \mathbb{R},\ P \in \mathbb{R}^{\mathbb{Z}}. \tag{5.2}$$

For a closer look at $L^d(t)P^*$ we note that $L^d(\mu_k)P^* = \sigma_k^2/(2d)$, and between each pair of Greville abscissae, it is a quadratic function with $D^2 L^d(t)P^* = 1$. Hence,

$$L^d(t)P^* = \frac{(\mu_k - t)\sigma_{k-1}^2 + (t-\mu_{k-1})\sigma_k^2}{2d(\mu_k - \mu_{k-1})} - \frac{(\mu_k - t)(t-\mu_{k-1})}{2}, \quad t \in [\mu_{k-1}, \mu_k].$$

A simplifying estimate for this expression can be found using a standard bound on the variance in terms of the range of data,

$$L^d(t)P^* \leqslant \frac{\max\{\sigma_{k-1}^2, \sigma_k^2\}}{2d} \leqslant \frac{(t_{k+d} - t_k)^2}{8(d-1)}, \quad t \in [\mu_{k-1}, \mu_k].$$

Using a standard stability argument, saying that the semi-norms $\|\Delta^2 P\|_\infty$ and $\|D^2 B^d P\|_\infty$ are equivalent, we easily recover the quadratic convergence of the control polygon to the spline under knot insertion as the local knot spacing tends to zero.

5.2. Global bounds

The pointwise estimate (5.2) implies the following global bounds.

Theorem 5.2. *Let $I \subseteq \mathbb{R}$ be an interval, $1 \leqslant p \leqslant \infty$, and P^* be defined according to* (5.1). *The L_p-norm on I is denoted by $\|\cdot\|_{I,p}$, and*

$$\|\Delta^2 P\|_{I,\infty} := \max_{t \in I} \|\Delta^2 P\|_{t,\infty}.$$

Then

$$\|L^d P\|_{I,p} \leqslant \|L^d P^*\|_{I,p} \|\Delta^2 P\|_{I,\infty}$$

for all $P \in \mathbb{R}^{\mathbb{Z}}$, where equality holds if $B^d P \in \mathbb{P}^2$.

It is straightforward to evaluate the norm $\|L^d P^*\|_{I,p}$ of the positive, piecewise quadratic function $L^d P^*$ provided that T, p, and d are explicitly known. More general results can be obtained for the following special cases.

First, let $p=\infty$. Since $L^dP^*=|L^dP^*|$ is a piecewise convex function, it attains its maximum either at a Greville abscissa or at the boundary of $I=[a,b]$,

$$\|L^dP^*\|_{I,\infty}=\max\bigl\{L^d(a)P^*, L^d(b)P^*, \sigma_j^2/(2d)\colon\ \mu_j\in I\bigr\}.$$

Second, we consider the important special case of integer knots, $T=\mathbb{Z}$. Here we have $\mu_j=j+(d+1)/2$ and $P_j^*=\mu_j^2/2-(d+1)/24$. The function L^dP^* is 1-periodic and formed by shifts of the function $f(t)=(d+1)/24-t(1-t)/2$, $t\in[0,1]$. For intervals of the form $I_k=[\mu_k,\mu_{k+1}]$ we obtain

$$\|L^dP^*\|_{I_k,p}=\|f\|_{[0,1],p},$$

for instance

$$\|L^dP^*\|_{I_k,1}=\frac{d-1}{24},\qquad \|L^dP^*\|_{I_k,2}=\left(\frac{5d^2-10d+9}{2880}\right)^{1/2},$$
$$\|L^dP^*\|_{I_k,\infty}=\frac{d+1}{24}.$$

Of course, it is also possible to recover the results of the second section on polynomials in Bernstein–Bézier form from Theorem 5.2 by choosing a knot sequence T with d-fold integer knots.

5.3. Tensor product splines

In complete analogy to the discussion of tensor product polynomials, bounds on tensor product splines can be obtained easily with the help of the triangle inequality. For the deviation

$$\mathbf{L}^{\mathbf{d}}(t_1,t_2)P:=\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}}B_{j_1}^{d_1}(t_1)B_{j_2}^{d_2}(t_1)P_{j_1,j_2}$$

we obtain

$$\bigl|\mathbf{L}^{\mathbf{d}}(t_1,t_2)P\bigr|\leqslant L^{d_1}(t_1)P^*\|\Delta_1^2P\|_{t_1,\infty}+L^{d_2}(t_2)P^*\|\Delta_2^2P\|_{t_2,\infty}.$$

On the interval $\mathbf{I}=I_1\times I_2$ the L_p-norm of the deviation is bounded by

$$\begin{aligned}\|\mathbf{L}^{\mathbf{d}}P\|_{\mathbf{I},p}&\leqslant\bigl\|L^{d_1}(t_1)P^*\|\Delta_1^2P\|_{I_1,\infty}+L^{d_2}(t_2)P^*\|\Delta_2^2P\|_{I_2,\infty}\bigr\|_{\mathbf{I},p}\\&\leqslant|I_2|^{1/p}\|L^{d_1}P^*\|_{I_1,p}\|\Delta_1^2P\|_{I_1,p}+|I_1|^{1/p}\|L^{d_2}P^*\|_{I_2,p}\|\Delta_2^2P\|_{I_2,p}.\end{aligned}$$

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